Quantum Frameness for Charge-Parity-Time inversion symmetry

Michael Skotiniotis,[1, 2, *] Ian T. Durham,[3, †] Borzu Toloui,[1, 4] and Barry C. Sanders[1]

[1]*Institute for Quantum Science & Technology, University of Calgary, Calgary, Alberta T2N 1N4, Canada*

[2]*Institut für Theoretische Physik, Universität Innsbruck, Technikerstr. 25, A-6020 Innsbruck, Austria*

[3]*Department of Physics, Saint Anselm College, Manchester, NH 03102*

[4]*Department of Physics, Haverford College, 370 Lancaster Avenue, Haverford, PA 19041, United States*

(Dated: January 4, 2013)

We develop a charge-parity-time (CPT) frameness resource theory to circumvent CPT-superselection. We construct and quantify such resources for spins 0, $\frac{1}{2}$, and 1 and for Majorana particles, and we show that spin 0 particles only admit classical information processing whereas particles of higher-dimensional spin permit quantum information processing in the presence of CPT-superselection. Our treatment of CPT inversion as indecomposable circumvents the anti-unitarity of certain actions (C and T) by strictly considering the aggregate CPT.

PACS numbers: 03.67.-a,03.67.Hk,11.30.Er,11.30.Fs

Superselection rules such as those for charge [1, 2], orientation [3], chirality [1, 4, 5], phase [2, 6, 7], and time [8] forbid certain coherent quantum superpositions of states and are formally equivalent to the lack of a requisite frame of reference [9]. Superselection is circumvented by consuming appropriate frameness resources, namely states that are asymmetric with respect to the group G of transformations [10]. Here we (i) introduce superselection for charge-parity-time (CPT) invariance [11–13], (ii) construct the CPT frameness resources for spins 0, $\frac{1}{2}$, and 1 and for Majorana particles, and (iii) quantify the frameness of such resources in an operational and information-theoretic manner. We consider CPT as an indecomposable symmetry with a unitary representation as opposed to a decomposable symmetry consisting of charge (C), parity (P), and time (T) inversions with a concomitant ill-defined anti-unitary representation for T [8], which can paradoxically lead to invariant states evolving to non-invariant resources under evolution generated by a Hamiltonian that commutes with every element of the representation of G. One important result is that CPT superselection must be circumvented in order to communicate quantum information with spin 0 particles whereas quantum communication is viable for higher-spin particles even with CPT superselection. Our new theory is fundamentally appealing in that it connects the quantum resource of frameness with the relativistic universe via fundamental CPT symmetry.

CPT symmetry concerns the simultaneous inversion of C, P, and T represented respectively by (italicized) operators C, P, and T. We avoid the difficulties arising from representations for anti-unitarity transformations by constructing a unitary representation for the action of CPT on the state space of a physical system using the Feynman-Stueckelberg interpretation of anti-matter as the CPT image of matter [14–17]. CPT inverts all conserved internal degrees of freedom. We define the total internal quantum number, $p := Q + (A - L)$, where Q is the total charge and $A - L$ is the conserved difference between baryon number A and lepton number L neither of which is universally conserved by itself. The CPT action on a particle with total internal quantum number p and spin s is

$$CPT\,|p,s\rangle = e^{i\theta}\,|-p,-s\rangle\,,\ \theta \in [0,2\pi). \tag{1}$$

As $(CPT)^2 = \mathbb{1}$ then $\theta \in [0,\pi)$ and $\{\mathbb{1}, CPT\}$ is a representation of $\mathbb{Z}_2$ (finite group of two elements). We now show that, for the cases of spins $s \in \{0, \frac{1}{2}, 1\}$ with either massive and massless particles, $\{\mathbb{1}, CPT\}$ is a unitary representation of $\mathbb{Z}_2$. We suggest a method by which higher-order s cases should follow similarly.

Consider a single, massive $s = 0$ particle with total internal quantum number p governed by the Klein-Gordon equation $[\Box - (mc/\hbar)^2]\psi = 0$ with $\Box$ the D'Alembertian operator [18]. In free space the Klein-Gordon equation solutions are positive and negative four-momentum plane waves $|\pm p\rangle$, which span a two-dimensional Hilbert space. The state $|-p\rangle$ has negative energy hence is associated with the anti-matter state.

From Eq. (1), $CPT = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with eigenstates

$$|\pm\rangle \equiv \frac{1}{\sqrt{2}}\left(|p\rangle \pm |-p\rangle\right) \tag{2}$$

corresponding to eigenvalues ± 1, and σ^1 is the first of three Pauli matrices given by the three-component vector of Pauli matrices $\boldsymbol{\sigma} := (\sigma^1, \sigma^2, \sigma^3)$. Explicit mention of the choice of p is dropped in the eigenstate notation in the left-hand side of Eq. (2). Thus we see that CPT is unitary for a massive, spin 0 particle. For the massless spin 0 particle, the Klein-Gordon equation is simply $\Box\psi = 0$, and the space is still spanned by $|\pm p\rangle$ so CPT is the same unitary operator as for the massive-particle case.

We now consider the representation of CPT for the case of a massive $s = \frac{1}{2}$ Dirac spinor with definite total internal symmetry p, i.e. a physical system satisfying the Dirac equation $\left(i\hbar\partial\!\!\!/ + mc\right)\psi = 0$ for $\partial\!\!\!/ = \gamma^\mu\partial_\mu$

with Einstein summation notation over $\mu \in \{0,1,2,3\}$, $\partial_\mu := \partial/\partial x^\mu$ and $\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$, $\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}$. For fixed four-momentum p, the solutions of the Dirac equation belong to a four-dimensional Hilbert space spanned by the orthonormal states

$$\left|p,\frac{1}{2}\right\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \left|p,-\frac{1}{2}\right\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},$$
$$\left|-p,\frac{1}{2}\right\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \left|-p,-\frac{1}{2}\right\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \qquad (3)$$

where, without loss of generality, we choose the basis that diagonalizes σ^3.

As CPT reverses all degrees of freedom, for the case of a massive Dirac spinor $CPT = \gamma^0\gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}$ with eigenvectors

$$|\pm,0\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|p,\frac{1}{2}\right\rangle \pm \left|-p,-\frac{1}{2}\right\rangle\right),$$
$$|\pm,1\rangle \equiv \frac{1}{\sqrt{2}}\left(\left|p,-\frac{1}{2}2\right\rangle \pm \left|-p,\frac{1}{2}\right\rangle\right), \qquad (4)$$

where the vectors $|\pm,0\rangle$, $|\pm,1\rangle$ are the two eigenstates with eigenvalue ± 1. Hence, unlike massive spin 0 particles, the unitary CPT operator for Dirac spinors has a two-fold degenerate spectrum for both ± 1 eigenvalues.

By definition, Dirac spinor states that are invariant under the action of the CPT operator are known as Majorana spinors [18, 19]. Majorana spinors are also invariant under this action when given as solutions to the Majorana equation $i\hbar\partial\!\!\!/\psi_c + mc\psi = 0$, where, in the Majorana basis, $\psi_c := i\psi^*$. For Majorana spinors, this Majorana equation is equivalent to the Dirac equation.

Massless spin $\frac{1}{2}$ particles are governed by the Weyl equation $i\hbar\sigma^\mu\partial_\mu\psi = 0$ for $\sigma^0 := \mathbb{1}$, and its solutions are Weyl spinors. Whereas these spinors are typically represented as two-component spinors, they are equivalent to four-component spinor representations wherein the Weyl equation is identical to the Dirac equation with $m \equiv 0$ [18]. In such a case, the solutions belong to a Hilbert space spanned by the same orthonormal states as those in Eq. (3). States of $+p$ are identified as right-handed and states of $-p$ are identified as left-handed. As such the CPT operator for spin $\frac{1}{2}$ particles is the same regardless of whether they are massive or massless.

For massive spin 1 particles, the Weinberg-Shay-Good equations [18]

$$\left[i\hbar\partial_\mu(\gamma^{\mu\nu} - g^{\mu\nu})i\hbar\partial_\nu + 2m_0^2c^2\right]\psi = 0 \qquad (5)$$

are appropriate with $\gamma^{\mu\nu}$ the 6×6 matrices

$$\gamma^{ij} = \begin{pmatrix} 0 & \delta_{ij}\mathbb{1} + M^{(ij)} + M^{(ji)} \\ \delta_{ij}\mathbb{1} + M^{(ij)} + M^{(ji)} & 0 \end{pmatrix}$$

and $\gamma^{0i} = \gamma^{i0} = \begin{pmatrix} 0 & S^i \\ -S^i & 0 \end{pmatrix}$, $\gamma^{00} = -\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$. Here

$$S^1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; S^2 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad (6)$$

and

$$S^3 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (7)$$

and also

$$M^{(ij)} = iS^j iS^i. \qquad (8)$$

The solutions to the Weinberg-Shay-Good equation belong to a six-dimensional Hilbert space spanned by the orthonormal states

$$|p,1\rangle = \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix} \; |p,0\rangle = \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix} \; |p,-1\rangle = \begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix},$$

$$|-p,1\rangle = \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}, \; |-p,0\rangle = \begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}, \; |-p,-1\rangle = \begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix} \qquad (9)$$

thereby yielding

$$CPT = i^2\gamma^{00} = \begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \end{pmatrix}. \qquad (10)$$

Note that Eq. (10) is not composed of the spin 1 versions of σ^1 though it is unitary with eigenvectors

$$|\pm,0\rangle = \frac{1}{\sqrt{2}}(|p,1\rangle \pm |-p,-1\rangle),$$
$$|\pm,1\rangle = \frac{1}{\sqrt{2}}(|p,0\rangle \pm |-p,0\rangle), \qquad (11)$$

and

$$|\pm,2\rangle = \frac{1}{\sqrt{2}}(|p,-1\rangle \pm |-p,1\rangle), \qquad (12)$$

where the vectors $|\pm,0\rangle, |\pm,1\rangle, |\pm,2\rangle$ are the three eigenstates of Eq. (10) with eigenvalue ±1.

Massless spin 1 particles, of which photons are the only known type, present a unique challenge. Prior to the development of the Białynicki-Birula—Sipe (BB-S) wavefunction for photons [20–23], they were either treated in a non-local manner [24–28] or, as Raymer and Smith point out, the photon-as-particle language was largely abused [29]. The BB-S results present a truly local single-photon wavefunction. The associated wave dynamics have since been extensively developed [29–31], and the wave equation is

$$i\hbar\left(\partial_0 + c\beta^3 S^j\partial_j\right)\psi = 0, \tag{13}$$

where S^j is given in Eqs. (6) and (7) and $\beta^3 = \begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}$. The solutions are six-component spinors where, in the Weyl represention, $\psi = \begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}$ in which Ψ_+ and Ψ_- represent opposite helicities [29]. Given that in free space the helicities are usually assumed not to mix, they are often treated separately [20, 29]. However, as we show, the eigenvectors of the CPT operator are superpositions of opposite helicities.

For the solutions of Eq. (13) to describe a photon correctly, we require the auxiliary condition $\psi = \beta^1\psi^*$ [20] for $\beta^1 = \begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}$. For the general case of massless spin 1 particles, the solution space of Eq. (13) is the same as for Eq. (5) and thus is spanned by the states given in Eq. (9). Consequently the CPT operator has the same form as that for the massive spin 1 particles given in Eq. (10). In terms of the β^j matrices, we can write $CPT = (\beta^3)^2\beta^1$. Thus the eigenvectors are those given by Eq. (12). We note, however, that the photon does not possess a state of zero total spin (corresponding to the absence of longitudinal polarization in classical optics) so the states $|p,0\rangle$ and $|-p,0\rangle$ are unphysical.

For particles of arbitrarily higher spin, solutions may be constructed from solutions to the Bargmann-Wigner equations which are individually indexed Dirac equations. For instance, spin $\frac{3}{2}$ particles obey a version of the Bargmann-Wigner equations known as the Rarita-Schwinger equation whose solutions are 16-component spinors or equivalently four 4-component spinors, each of which is individually a solution of the Dirac equation.

We are now ready to formulate a CPT superselection rule. Due to Schur's lemmas [32], unitary representations of finite groups can be fully reduced into their irreducible representations (irreps). Furthermore, all inequivalent irreps of abelian groups are one-dimensional, and in particular the irreps of $\mathbb{Z}_2$ are given by ±1. As CPT superselection implies that coherent superspositions between eigenstates of the CPT operator cannot be observed [10], the Hilbert space of any system subject to CPT superselection may be conveniently written as

$$\mathscr{H} \cong \bigoplus_{p=\pm1}\mathscr{H}^{(p)}, \tag{14}$$

where $p=\pm1$ denotes the two inequivalent irreps of $\mathbb{Z}_2$, and $\mathscr{H}^{(p)}$ are the corresponding eigenspaces. For spin 0 particles, $\mathscr{H}^{(1)}$ is spanned by $|+\rangle$ of Eq. (2), whereas $\mathscr{H}^{(-1)}$ is spanned by $|-\rangle$ of Eq. (2). For Dirac spinors $\mathscr{H}^{(1)}$ is the two-dimensional space spanned by the states $\{|+,0\rangle, |+,1\rangle\}$ of Eq. (4), whereas $\mathscr{H}^{(-1)}$ is spanned by the states $\{|-,0\rangle, |-,1\rangle\}$ of Eq. (4). Similarly, for spin 1 particles $\mathscr{H}^{(1)}$ is the three-dimensional space spanned by the states $\{|+,0\rangle, |+,1\rangle, |+,2\rangle\}$ of Eq. (12), whereas $\mathscr{H}^{(-1)}$ is spanned by the states $\{|-,0\rangle, |-,1\rangle, |-,2\rangle\}$ of Eq. (12).

Hence, a party subject to a CPT superselection rule is equivalent to a party lacking a reference frame for CPT, and this lack of a reference frame imposes restrictions on the types of states and the types of operations that can be performed. In the absence of a CPT frame of reference a party can only perform CPT-invariant operations and prepare CPT-invariant states. These CPT-invariant operations are of two types: shifts in the irrep label p and changes to the relative amplitudes of different eigenstates of p [10]. On the other hand, CPT-invariant states belong in either $\mathscr{H}^{(1)}$ or $\mathscr{H}^{(-1)}$. Thus, a linear superposition of eigenstates of CPT is a resource and can be brought, via CPT-invariant operations, to the standard form

$$|q\rangle := \sqrt{q}\,|+\rangle + \sqrt{1-q}\,|-\rangle\,,\ q\in[0,1]. \tag{15}$$

For the case of higher-dimensional spinors one can pick without loss of generality one state from each eigenspace, and can bring any resource state via CPT-invariant operations to the form of Eq. (15). Note also that, for the case of photons and other massless spin 1 particles, this implies that the resource states include superpositions of states of opposite helicity. This is contrary to the usual assumption, stated earlier, that in free space the helicities are assumed *not* to mix [20, 29]. We now discuss CPT resources operationally.

Consider two parties, Alice and Bob, who share all knowledge regarding all other frames of reference, such as phase, orientation etc., but lack a shared frame of reference for CPT. Alice and Bob require a shared CPT frame of reference. To that end Alice prepares a quantum system in one of the states $\{|q\rangle, CPT\,|q\rangle\}$ and sends this state to Bob who performs a measurement and attempts to infer which one of the two states Alice has sent him.

As Alice and Bob are completely ignorant of each other's CPT frame, let X denote the random variable corresponding to Alice's CPT frame of reference, consisting of the elements of $\mathbb{Z}_2$ each with equal probability and Y the random variable corresponding to Bob's measurement outcome. The frameness inherent in Alice's

CPT reference frame token is quantified by the alignment rate R, which, for the case of reference frames associated with unitary representations of $\mathbb{Z}_2$, was shown to be [33] $R(q) = -2\log|2q-1|$. The alignment rate quantifies the amount of information Bob acquires via measurement per copy of the resource state, $|q\rangle$, corresponding to Alice's reference frame, in the limit of asymptotically many copies [33]. Thus, if $q=\frac{1}{2}$, R is infinite because such states, corresponding to charged mesons, electrons, positrons, or similar particles, are completely asymmetric with respect to the CPT operator; they are *perfect* resources for a CPT frame of reference.

In closing, we prove that the restrictions imposed by CPT superselection must be lifted in order to communicate quantum information with spin 0 particles, whereas quantum information may be performed with higher-dimensional spinors even in the presence of CPT superselection. First consider the case of spin 0 particles, and let Alice prepare the quantum state $|\psi\rangle = c_0\,|+\rangle + c_1\,|-\rangle$, $c_0,\, c_1 \in \mathbb{C}$. Bob describes this state as

$$\begin{aligned}\rho &= \frac{1}{2}\left[|\psi\rangle\langle\psi| + CPT|\psi\rangle\langle\psi|(CPT)^{\dagger}\right]\\ &= |c_0|^2\,|+\rangle\langle+| + |c_1|^2\,|-\rangle\langle-|.\end{aligned} \tag{16}$$

Hence, complete lack of a shared CPT frame of reference destroys all coherence in the state of a spin 0 particle. Consequently, Alice and Bob must necessarily lift the restrictions imposed by CPT superselection in order to communicate quantum information using spin 0 particles.

Contradistinctively, Alice and Bob can communicate quantum information in the presence of CPT superselection using higher-dimensional spinors. As an example, consider the case of Dirac spinors where Alice prepares the state $|\psi\rangle = c_0\,|+,0\rangle + c_1\,|+,1\rangle$, which is invariant under CPT. Consequently, Bob describes this state the same way as Alice. Thus, coherences between $|+,0\rangle$ and $|+,1\rangle$ are preserved under CPT superselection, and Alice and Bob can use a single fermion or anti-fermion to perform quantum computation.

In summary we have shown that the superselection rule arising from CPT symmetry can be circumvented using CPT frameness resources. We have identified the ultimate frameness resources for the cases of both massive and massless spins 0, $\frac{1}{2}$, and 1 particles including Majorana spinors, and have suggested a path for finding solutions for states of arbitrary spin. We have also shown that CPT superselection must be lifted in order to perform quantum computation with spin-0 particles whereas for higher-dimensional spins quantum computation can be performed even in the presence of CPT superselection. Our theory connects quantum resources and superselection to the most fundamental symmetry of relativistic quantum physics.

We acknowledge the following financial support for this project: NSERC, USARO, and FWF Grants Y535-N16, P20748-N16, P24273-N16, and F40-FoQus F4011/12-N16 for MS; MITACS and NSERC for BT; FQXi for ITD; and NSERC, MITACS, AITF, and CIFAR for BCS. We appreciate valuable discussions with I. Marvian and Y. R. Sanders.

* michael.skotiniotis@uibk.ac.at
† idurham@anselm.edu

[1] G. C. Wick, A. S. Wightman, and E. P. Wigner, Phys. Rev. **88**, 101 (1952).
[2] Y. Aharonov and L. Susskind, Phys. Rev. **155**, 1428 (1967).
[3] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, Phys. Rev. Lett. **91**, 027901 (2003).
[4] D. Collins, L. Diosi, N. Gisin, S. Massar, and S. Popescu, Phys. Rev. A **72**, 022304 (2005).
[5] N. Gisin, arXiv:quant-ph/0408095 (2004).
[6] T. Rudolph and B. C. Sanders, Phys. Rev. Lett. **87**, 077903 (2001).
[7] B. C. Sanders, S. D. Bartlett, T. Rudolph, and P. L. Knight, Phys. Rev. A **68**, 042329 (2003).
[8] G. Gour, B. C. Sanders, and P. S. Turner, J. Math. Phys. **50**, 102105 (2009).
[9] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, Rev. Mod. Phys. **79**, 555 (2007).
[10] G. Gour and R. W. Spekkens, New J. Phys. **10**, 033023 (2008).
[11] J. Schwinger, Phys. Rev. **82**, 914 (1951).
[12] G. Lüders, Kongelige Danske Vidensabernes Selskab Matematisk-fysiske Meddelelser **28**, 1 (1954).
[13] W. Pauli, *Niels Bohr and the Development of Physics* (Pergamon Press, New York, 1955).
[14] E. C. G. Stueckelberg, Helv. Phys. Acta **15**, 42 (1942).
[15] R. P. Feynman, Phys. Rev. **74**, 939 (1948).
[16] R. P. Feynman, Phys. Rev. **76**, 749 (1949).
[17] R. P. Feynman, Phys. Rev. **76**, 769 (1949).
[18] W. Greiner, *Relativistic Quantum Mechanics Wave Equations, Third edition* (Springer-Verlag, Heidelberg, 2000).
[19] A. Duncan, *The Conceptual Framework of Quantum Field Theory* (Oxford University Press, Oxford, 2012).
[20] I. Bialynicki-Birula, Acta Physica Polonica A **86**, 97 (1994).
[21] I. Bialynicki-Birula, in *Coherence and Quantum Optics VII*, edited by J. Eberly, L. Mandel, and E. Wolf (Plenum, New York, 1995), p. 313.
[22] I. Bialynicki-Birula, in *Progress in Optics XXXVI*, edited by E. Wolf (Elsevier, Amsterdam, 1996), p. 245.
[23] J. E. Sipe, Phys. Rev. A **52**, 1875 (1995).
[24] L. Landau and R. Peierls, Zeitschrift für Physick **62**, 188 (1930).
[25] T. Newton and E. Wigner, Rev. Mod. Phys. **21**, 400 (1949).
[26] A. Wightman, Rev. Mod. Phys. **34**, 845 (1962).
[27] R. Cook, Phys. Rev. A **25**, 2164 (1982).
[28] R. Cook, Phys. Rev. A **26**, 2754 (1982).
[29] B. J. Smith and M. Raymer, New Journal of Physics **9**, 414 (2007).
[30] D. Kobe, Foundations of Physics **29**, 1203 (1999).
[31] M. Raymer and B. J. Smith, in *The Nature of*

Light: What is a Photon?, edited by C. Roychoudhuri, A. Kracklauer, and K. Creath (CRC Press, Boca Raton, 2008), pp. 207–214.
[32] J.-Q. Chen, *Group Representation Theory for Physicists* (World Scientific, Singapore, 1989).
[33] M. Skotiniotis and G. Gour, New J. Phys. **14**, 073022 (2012).